"CHESSBOARD" DIFFERENCE METHOD OF SOLVING A SYSTEM OF DIFFERENTIAL EQUATIONS OF HEAT AND MASS TRANSFER

A. F. Klement'ev and O. N. Balykina

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An explicit, absolutely stable difference method of solving a system of differential equations of heat and mass transfer is proposed. Limits of applicability of the method are estimated.

As we know [1], the system of differential equations of energy and mass transfer is written in the form

$$\frac{dv_i}{dt} = \frac{dv_i}{dt} + U\Delta v_i = \sum_{n=1}^{m} \nabla (R_{in} \nabla v_{in}) + \Pi_i \ (i,\ n = \overline{1,\ m}), \tag{1}$$

where Π_i is the amount of heat (substance) emitted by the sources in unit volume per unit time; R_{in} are thermophysical transfer coefficients between which there is no reciprocity relation, $R_{in} \neq R_{ni}$; U is the velocity vector; v_i are the sought functions; ∇ is the Hamilton operator.

Exact solution of the system (1) is difficult even for constant thermophysical characteristics. In [2, 3] an explicit difference method was proposed for a simplified system, while in [4] it was proposed for the complete system of equations (1). In [5] implicit difference schemes are set up for numerical solution of the problem of diffusion in a two-phase medium with a given internal (moving) boundary of phase transformation.

In the present work we consider an explicit difference method which is absolutely stable for a certain class of systems of the form (1).

We shall consider the one-dimensional problem for two substances of a fixed system with constant thermophysical characteristics. Then, mathematically, the problem is formulated as follows: find the function of concentration of two substances – the heat v(x, t) and the mass u(x, t) – satisfying within the region $\{a \le x \le b,\ t \ge 0\}$ the system of equations

$$\frac{\partial u}{\partial t} = R_{11} \frac{\partial^2 u}{\partial x^2} + R_{12} \frac{\partial^2 v}{\partial x^2}, \tag{2}$$

$$\frac{\partial v}{\partial t} = R_{21} \frac{\partial^2 u}{\partial x^2} + R_{22} \frac{\partial^2 v}{\partial x^2}, \tag{3}$$

with the matrix $\|R_{in}\|$, i, n = 1, 2, being positive-definite. On the boundary Γ of the region the conditions

$$u|_\Gamma = \varphi_1(t),\ v|_\Gamma = \varphi_2(t) \tag{4}$$

are specified.

At the instant of time t = 0 the initial distributions of heat and mass are given:

$$\begin{aligned} u|_{t=0} &= f_1(x), \\ v|_{t=0} &= f_2(x). \end{aligned} \tag{5}$$

We introduce the difference grid $\{t_k = k\tau,\ k = 0,\ 1,\ 2,\ \ldots;\ x_j = a + jh,\ j = \overline{0,\ N},\ N = (b-a)/h\}$ and denote the approximate values of u and v at the point $(x_j,\ t_k)$ by u_j^k, v_j^k, respectively. We divide the set of points of the

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TABLE 1. Calculation Parameters

Variants	1	2	3	4	5	6	7	8	9	10	11
R_{11}	5	5	6	3	3	5	2	2	1	4	3
R_{12}	0	0	0	0	3	1	3	1	1	1	1
R_{21}	6	6	0	2	2	1	2	1,5	3	2	2
R_{22}	5	4	5	3	2	4	1	2	2	2	1

Variants	12	13	14	15	16	17	18	19	20
R_{11}	3	3	2	4	5	4	2	4	3
R_{12}	1	1	1	1	1	0,5	1	1	3
R_{21}	1,5	3	3	1	1	2	4	1	1
R_{22}	2	2	2	2	2	2	2	1	3

grid into two subsets: points whose sum of indices j+k is even we call "explicit," while points whose sum of indices is odd we call "implicit." It is obvious that each explicit point of the plane (x, t) is surrounded by implicit points, and conversely.

At explicit points we calculate the system of equations (1) by means of the explicit difference scheme

$$
\begin{aligned}
u_j^{k+1} &= P_j^k + (1 - 2r_{11})\, u_j^k, \quad v_j^{k+1} = Q_j^k + (1 - 2r_{22})\, v_j^k, \\
P_j^k &= r_{11}(u_{j\pm 1}^k) + r_{12}(v_{j\pm 1}^k - 2v_j^k), \\
Q_j^k &= r_{21}(u_{j\pm 1}^k - 2u_j^k) + r_{22}(v_{j\pm 1}^k), \\
u_{j\pm 1}^k &= u_{j+1}^k + u_{j-1}^k, \quad r_{in} = \frac{\tau}{h^2} R_{in}.
\end{aligned} \tag{6}
$$

At implicit points the calculation is carried out according to the implicit difference scheme extended relative to u_j^{k+1}, v_j^{k+1}:

$$
\begin{aligned}
u_j^{k+1} &= (1 + 2r_{11})^{-1}(P_j^{k+1} + u_j^k), \\
v_j^{k+1} &= (1 + 2r_{22})^{-1}(Q_j^{k+1} + v_j^k).
\end{aligned} \tag{7}
$$

We shall calculate the values of the functions u_j^{k+1}, v_j^{k+1} in a "chessboard" order: first, by the recursive expressions (6) we find these values at explicit points. Then, substituting the values of heat and mass just found at the explicit points of the (k+1)-th layer into the expressions (7), we find the sought values at the implicit points.

It is natural to call the method described by the expressions (6) and (7) the "chessboard" method [6, 7]. It approximates the system (1) with an error of the order $O[h^2 + \tau^2 + (\tau^2/h^2)]$.

We should note the following advantages of the "chessboard" difference method over the classical difference schemes.

1. A reduction of the amount of computations approximately 10-15 times in comparison with the explicit difference scheme, and 2-3 times in comparison with the implicit difference scheme.

2. A reduction of the required volume of operative memory about 2 times.

3. The ease of application for complex problems. The "chessboard" without any alterations is transferred to the system (1) of any dimensionality with variable coefficients.

In the case of the dependence of the thermophysical characteristics on the sought functions, Eqs. (6) and (7) remain explicit if the arguments of the functions R_{in} at implicit points are interpolated from explicit points.

We proceed to discover the limits of applicability of the "chessboard" method. The stability of the method on the set of implicit points gives rise to no doubts, and it remains for us to verify its stability on the set of explicit points.

Excluding from Eqs. (6) the values of unknown functions at implicit points by means of Eqs. (7), it is easy to show that the "chessboard" method on the set of explicit points is equivalent to the difference scheme of Dewfort–Frankel [8]

$$
\begin{gathered}
u_j^{k+1} = (1 + 2r_{11})^{-1}[2r_{11}(u_{j\pm 1}^k) + (1 - 2r_{11})U_j^k + 2r_{12}(v_{j\pm 1}^k - 2v_j^k)],\\
U_j^{k+1} = u_j^k,\\
v_j^{k+1} = (1 + 2r_{22})^{-1}[2r_{21}(u_{j\pm 1}^k - 2u_j^k) + 2r_{22}(v_{j\pm 1}^k) + (1 - 2r_{22})V_j^k],\\
V_j^{k+1} = v_j^k.
\end{gathered}
\tag{8}
$$

Substituting the Fourier integrals into (8), we obtain the transition matrix [8]

$$
G = \begin{bmatrix}
2\dfrac{\alpha_1}{\beta_1}x & \dfrac{1-\alpha_1}{\beta_1} & -4\dfrac{r_{12}}{\beta_1}(1-x) & 0\\
1 & 0 & 0 & 0\\
4\dfrac{r_{21}}{\beta_2}(1-x) & 0 & 2\dfrac{\alpha_2}{\beta_2}x & \dfrac{1-\alpha_2}{\beta_2}\\
0 & 0 & 1 & 0
\end{bmatrix},
$$

where we have denoted

$$\alpha_i = 2r_{ii},\ \beta_i = 1 + \alpha_1,\ i = \overline{1,\ 2};\ x = \cos\theta.$$

We know that [8] for the stability of the scheme (8) it is necessary and sufficient for the elements of the matrix $G(t, \theta)$ to be uniformly bounded for all $0 < t < T$, $|\theta| < \pi$ and all eigenvalues λ_m of the matrix, with an exception, perhaps, of one (for example, $|\lambda_1| \leq 1 + 0(\tau)$) to lie strictly within a unit circle: $|\lambda_m| < 1$.

The characteristic equation for G has the form

$$
\begin{gathered}
F(\lambda) = b_0\lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4 = 0,\\
b_0 = 1 + \alpha + \frac{\rho}{2},\ b_1 = -x(\rho + 4\alpha),\\
b_2 = 2[2(\alpha - \beta)x^2 + 4\beta x + 2\beta + \alpha - 1],\\
b_3 = x(\rho - 4\alpha),\ b_4 = 1 + \alpha - \frac{\rho}{2},\\
\rho = 2(\alpha_1 + \alpha_2),\ \alpha = \alpha_1\alpha_2,\ \beta = r_{12}r_{21}.
\end{gathered}
$$

Applying to the polynomial $F(\lambda)$, transformed by the substitution $\lambda = (\mu + 1)/(\mu - 1)$, the Routh–Hurwitz criterion, we find that the Hurwitz conditions are fulfilled for

$$\alpha > \beta,\ \rho > 0,\ \beta = 0. \tag{9}$$

Thus the "chessboard" method on a set of explicit points is absolutely stable for systems (2), (3) for which the first two conditions, obviously, are satisfied with respect to the physical sense of the problem. The third condition considerably narrows the class of problems being considered. However, it is fulfilled in many cases of practical importance [2, 3, 9-12].

Thus, when the conditions (9) are fulfilled, stability of this method holds for p-dimensional systems of the form (2), (3), since the expressions (6), (7) preserve their form (only the numerical coefficients of j_j^k, v_j^k and in P_j^k, Q_j^k are altered).

To verify the stability conditions, calculations of a series of systems (2), (3) were carried out on a BÉSM-4 digital computer for the following values of parameters and boundary and initial conditions:

$$
\begin{gathered}
a = 0,\ b = 1,\ h = 0,\ 1,\ \tau = rh^2,\ f_1 = f_2 = 1,\\
u(0,\ t) = u(1,\ t) = 0,\ v(0,\ t) = v(1,\ t) = 0.
\end{gathered}
$$

Out of the 20 values of R_{in} presented in Table 1, the conditions (9) have been fulfilled only for the first four variants. However, the calculations show that the "chessboard" scheme is absolutely stable for the variants 1-4, in the case 5 instability appears for $r > 2$, for 6-9 it appears for $r > 1$, for 10 it appears for $r > 0.5$, and only for the variant 20 is the scheme absolutely unstable. In [4] results of investigating a classical explicit scheme for the given problem for $r = 0.1$ are presented; here instability was discovered for the variants 3-4, 6-7, 9, 16, 17, while by the "chessboard" scheme, for $r = 0.1$, only one variant does not pass.

Thus, the limits of applicability for problems of heat and mass transfer of the "chessboard" method are broader than follows from the stability conditions (9).

NOTATION

R_{in}, thermophysical transfer coefficients; R_{11}, coefficient of thermal diffusivity; R_{22}, diffusion coefficient; R_{12}, mass diffusion coefficient; R_{21}, thermal diffusion coefficient; v(x, t) heat function; u(x, t) mass function; h, τ, grid pitches; u_j^k, v_j^k grid analogs of the functions u, v; G, transition matrix; λ_m, eigenvalues of the matrix G; $F(\lambda)$, characteristic polynomial of the matrix G.

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SOLUTION OF THE TWO-DIMENSIONAL UNSTEADY DIFFUSION EQUATION FOR VORTEX FLOW

M. A. Puzrin, O. M. Todes, and M. Z. Fainitskii

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A numerical method of solution based on the use of probability analogies is presented. An example of a calculation by the scheme developed is given.

Solid particles in fluidized bed devices take part in both random and directed motions in the form of circulating flows through the whole reactor [1, 2]. This circulation can be represented as a vortex superimposed on the diffusion intermixing of the solid phase. The intermixing process must then be described by an inhomogeneous differential equation for diffusion in vortex flow. It is very difficult or impossible to obtain an analytic solution of this equation. The method of finite differences is a universal method for obtaining approximate solutions of differential equations and is applicable to a broad class of problems [3].

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